Problem 1. Use the Master theorem to give upper bounds for the following functions.

a) \( f(n) = f(n/4) + 2n^{1/2} \)

b) \( f(n) = 3f(n/4) + 2n^{1/2} \)

Problem 2. Consider the algorithm below which computes \( \lfloor a^{1/b} \rfloor \) where \( a \) and \( b \) are positive integers.

```plaintext
function search(a, b, l, r)
    if l = r then
        return l
    end if
    m ← \( \lceil (l+r)/2 \rceil \)
    if \( m^b \leq a \) then
        return search(a, b, m, r)
    else
        return search(a, b, l, m - 1)
    end if
end function

function root(a, b)
    return search(a, b, 1, a)
end function
```

a) List all calls to `search` (direct or indirect) that `root(150, 3)` makes. How many are there?

b) Use the Master theorem to upper bound the running time of `root(a, b)`. Assume \( a \) is a power of 2.

Problem 3.

a) Convert 303 to binary.

b) Convert 303 to base 3.

c) Convert 1001011₂ to decimal.

d) Convert 12012₃ to decimal.

e) What is the last digit of 123456789012345678 × 609 in base 6. Don’t try to compute the product.

Problem 4. Let \( f_n \) be Fibonacci numbers with \( f_0 = 0, f_1 = 1 \). This problem asks you to prove that any positive integer can be uniquely represented as a sum of Fibonacci numbers \( f_2, f_3, f_4, \ldots \) with no repetitions and no two consecutive Fibonacci numbers. Note that \( f_0 \) and \( f_1 \) cannot be used.

For instance, \( 20 = f_7 + f_5 + f_3 \).

Thus, Fibonacci numbers form a peculiar number system.
a) Represent the following numbers as explained above: 1, 7, 14, 88, 199.

b) Prove using induction that any positive integer can be represented as a sum of Fibonacci numbers \( f_2, f_3, f_4, \ldots \), possibly with repetitions or consecutive Fibonacci numbers.

c) Now prove using strong induction that any positive integer can be represented as a sum of Fibonacci numbers \( f_2, f_3, f_4, \ldots \), with no consecutive Fibonacci numbers and no repetitions.

d) (Extra credit for 15%) Prove that for any positive integer its representation as a sum of Fibonacci numbers \( f_2, f_3, f_4, \ldots \) with no consecutive Fibonacci numbers and no repetitions is unique.

**Problem 5.** Recall that for non-negative integers \( a \) and \( b \), \( \gcd(a, b) \) is the largest integer that divides both \( a \) and \( b \). For instance, \( \gcd(108, 60) = 12 \) since \( 12 | 108 \) and \( 12 | 60 \) and there is no larger integer that divides both. To see this, we can perform the algorithm that humans most like: decompose \( a \) and \( b \) into products of primes and pick as many primes from both as possible. For instance,

\[
\begin{align*}
108 &= 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \\
60 &= 2 \cdot 2 \cdot 3 \cdot 5
\end{align*}
\]

We can pick at most two 2’s and one 3 from both. \( \gcd(108, 60) = 2 \cdot 2 \cdot 3 = 12 \).

This algorithm is convenient and works well for small numbers, but factoring is very inefficient when large numbers are involved. Below is an implementation of Euclid’s algorithm for \( \gcd \) which needs fewer steps on large inputs.

```plaintext
function euclid(a, b)
    if a = 0 then
        return b
    end if
    if b = 0 then
        return a
    end if
    if a \leq b then
        return euclid(a, b \mod a)
    else
        return euclid(a \mod b, b)
    end if
end function
```

a) Run `euclid(108, 60)` and list all recursive calls.

b) Prove that if \( a \) and \( b \) are integers such that not both are 0, then \( \gcd(a, b) = \gcd(a, b - a) \).

As a consequence, \( \gcd(a, b) = \gcd(a, b \mod a) \) if \( a \neq 0 \). To see this one can use the fact that \( \gcd(a, b) = \gcd(a, b - ak) \) for any integer \( k \) which can be proven the same way the claim in the previous part is proven. Now, \( b \mod a \) is simply \( b - ak \) for some integer \( k \). So, \( \gcd(a, b \mod a) = \gcd(a, b - ak) = \gcd(a, b) \).

c) Prove correctness of `euclid`. That is, show that

\[
\forall a, b \geq 0 \; (\neg(a = 0 \land b = 0) \rightarrow (\text{euclid}(a, b) = \gcd(a, b))).
\]
To do that, for all $n \geq 0$ define predicate $P(n)$ to be true if and only if

$$\forall a, b \geq 0 \ (\neg (a = 0 \land b = 0) \land a \leq n \land b \leq n \to (\text{EUCLID}(a, b) = \text{gcd}(a, b))).$$

Prove by induction that $P(n)$ is true for all $n \geq 0$ and conclude the proof.