CS591 - Tsourkakis

Graph Compression

for Distances

Graph Spanners

Readings

1) Stefano Leucci's Notes
   (This is the set of notes we'll go through)

2) Awerbuch - Peleg 1992

3) Althofer - Das - Dobkin - Joseph - Soares '93

For more (optional readings) see web page
Let $G$ be a graph with $n$ vertices and $m$ edges.

$$d_H(s,t) \leq \alpha d_G(s,t) + \beta \quad \forall s,t \in V(G).$$

Then $H$ is an $(\alpha, \beta)$-spanner of $G$.

**Lemma** Let $H$ be a graph with $n$ vertices and $m$ edges.

If $\text{girth}(H) \geq g = 2k+4$, then $m \leq \frac{n^{k+1}}{k}$.

**Proof** Suppose we iteratively remove nodes with degree $\leq m$.

We are left with a $(1/n^k)$-core of $H$. Call
If we prove \( \overline{F} \) is empty then we are done.

Suppose for the sake of contradiction this is not the case.

Take node \( v \in \text{reach}(H) \), and consider all paths of distance \( \leq k \) rooted at \( n \).

The graph \( T_r \) is a tree. (girth \( \geq 2k + 1 \),

longest possible distance between any two nodes \( u, v \leq 2k \) assuming \( m \))

\[
\left( \frac{m^k}{k} \right) (\frac{m}{k} + 1) (\frac{m}{k})^{k-1} = \frac{m^k}{k} + m^{k-1} \geq n.
\]

Contradiction: \( \overline{F} \) is empty!
Back to Spanners

Lemma: Let $G$ be a graph with girth $g$ and $\Omega(n^{1+k})$ edges.

OBS. 1. Let $G$ be such a graph. Then its $(2k+1)$-spanner $H$ is $G$.

Proof: Suppose not wlog. Let $e \in E(G) \setminus E(H)$.

\[ \exists \text{ cycle of length } \leq 2k \text{ in } G. \]

But girth $(G) \geq 2k+1.$
Additive Spanners $(1, 2)$ due to Aingworth

\[ w_p = 1 - \frac{1}{m^2}, \quad \text{each vertex in } \mathcal{A} \text{ has a neighbor in } \mathcal{S}. \]

Let \( P \) be the probability of selecting a vertex \( u \) such that none of its neighbors in \( \mathcal{A} \) are selected. Then

\[
0 < (1 - p) \deg(u) < e^{-p \deg(u)}
\]

\[
< e^{e p \left(-\frac{3 \log n}{\sqrt{n}}\right)} = \frac{1}{n^3}
\]

\[
\Pr(\exists u \in \mathcal{A} : \text{no neighbor in } \mathcal{S}) \leq |\mathcal{S}| \cdot \frac{1}{n^3} < \frac{1}{n^2}
\]

Hence, \( w_p > 1 - \frac{1}{n^2} \) every heavy vertex has a neighbor in \( \mathcal{S} \).

Lemma 2

\[ \# \text{ light edges } \leq n^{3/2}. \]
Algorithm

\[ H = \bigcup_{v \in S} \text{BFS-tree}(v) \cup \text{light edges}. \]

Claim \( H \) is \((1,2)\) spanner.

\[ d_H(v,t) = d_Q(v,t). \]

\[ d_H(s,t) = d_H(s,u) + d_H(u,t). \]

\[ d_H(s,t) \leq d_Q(s,u) + 1 + d_H(u,t) \text{ for some } u \in S. \]
\[
\leq d_G (x, u) + 1 + (1 + d_G (u, v)) = d_G (x, v) + 2
\]