Theorem (2^{ND} ORDER SUFFICIENT CONDITIONS).

$f: \mathbb{R}^n \to \mathbb{R}$ twice differentiable function. For some $x^* \in \mathbb{R}^n$ we have:

(1) $\nabla f(x^*) = 0$.

(2) $H_f(x^*) = \left.\frac{\partial^2 f}{\partial x^2}\right|_{x=x^*}$ is positive definite.

(the only difference from the previous lecture where we had the necessary condition is the strict positivity of the eigenvalues)

Then x^* is a strict local minimum.

Proof (once again based on Taylor's theorem)

Since the Hessian is positive definite, there exists $m>0$ such that $\forall \in \mathbb{R}^n$ $h^T H_f(x^*) h \geq m \|h\|^2$. By Taylor's theorem:

$$f(x^*+h) - f(x^*) = (\cancel{\nabla f(x^*)})^{\overset{0}{}}_{1\times n} \cdot \underset{n\times 1}{h} + \frac{1}{2} h^T H_f(x^*) h + o(\|h\|^2).$$

$$\geq \frac{m}{2}\|h\|^2 + o(\|h\|^2) > 0 \quad \text{for sufficiently small } \|h\|.$$

Thus, x^* is a strict local min. qed

Exercise: Prove that if $\nabla f(x^*) = 0$, $H_f(x^*)$ is negative definite then x^* is a local maximum.

Summary

(1) $\nabla f(x^*) = 0$, $\frac{\partial^2 f(x^*)}{\partial x^2} \geq 0 \Rightarrow x^*$: local min., point of inflection or saddle point.

(2) $\nabla f(x^*) = 0$, $\frac{\partial^2 f(x^*)}{\partial x^2} > 0 \Rightarrow x^*$: strict local min.

(3) $\nabla f(x^*) = 0$, $\frac{\partial^2 f(x^*)}{\partial x^2} \leq 0 \Rightarrow x^*$ local max, point of inflection or saddle point.

(4) $\nabla f(x^*) = 0$, $\frac{\partial^2 f(x^*)}{\partial x^2} < 0 \Rightarrow x^*$: strict local max.

(5) $\nabla f(x^*) = 0$, but Hessian has both positive and neg eigenvals

$\Longrightarrow$ x^* saddle point.

(e.g., $f(x,y) = x^2 - y^2$, $H_f\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$).